NCUMC-2014. Problems

1.Can one find real functions $f, g \in C^1(-1,1)$ such that

$$\begin{vmatrix} \int_{-1}^{1} f^2 dx & \int_{-1}^{1} f\, g dx \\ \int_{-1}^{1} f\, g dx & \int_{-1}^{1} g^2 dx \end{vmatrix} \neq 0, \quad \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \equiv 0 ?$$

2. Sequences $\{a_n\}$, $\{b_n\}$ are not convergent, but sequences $\{a_n + b_n\}$, $\{a_n b_n\}$ are convergent. Prove that the sequences $\{a_n\}$, $\{b_n\}$ have identical sets of condensation points (partial limits), and the set consists of two points.

3. Given positive integers m and n. Consider all $n \times m$ real matrices of rank at most 2 without zero entries. For any such matrix A consider an $n \times m$ sign matrix defined by $A'_{i,j} = sign(A_{i,j})$ (where $sign(x) = x / |x|$ for non-zero real x). Prove that the number of different sign matrices does not exceed $(m+n)^{m+n}$.

4. Find all functions $f(x): (0,\infty) \to (0,\infty)$ satisfying

$$\frac{1}{1+x+f(y)} + \frac{1}{1+y+f(z)} + \frac{1}{1+z+f(x)} = 1$$

whenever x, y, z are positive numbers and $xyz = 1$.

5. Given positive integers n. The polynomial $f(x)$ of degree $2n-1$ is so that $f - f^2$ is divisible by $x^n(1-x)^n$. Find all possible values of the leading coefficient of f .

6. Prove that there exists integer r such that hundred of initial digits of the number e^r coincide with the hundred of initial digits of the number π.

7. Let $q(x)$ be continuous bounded from below, i.e. there exists constant $c > 0$ such that $q(x) > -c$ for all x, and $\lim\limits_{x\to\infty} \int_x^{x+\omega} q(x)dx = \infty$ for any $\omega > 0$. Prove that for any fixed λ any non-trivial solution of the equation $y'' + \lambda - q(x) \; y = 0$ has finite number of roots at $(0,\infty)$.