Problem 3 Given positive integers n and m. Consider all $n \times m$ real matrices of rank at most 2 without zero entries. For any such matrix A consider an $n \times m$ *sign matrix* A' defined by $A'_{i,j} = \mathrm{sign(A_{i,j})}$ (where $\mathrm{sign(x) = x/|x|}$ for non-zero real x). Prove that the number of different sign matrices does not exceed $(m+n)^{m+n}$.

Solution. We may suppose that $m \leq n$. If $m = 1$ then evidently number of sign matrices is no more than $2^n \leq (m+n)^{m+n}$. Let $m \geq 2$ and $v_1, \ldots, v_m$ be columns of our matrix A of rank at most 2. Without loss of generality, no two of vectors $v_1, \ldots, v_m$ are collinear (and so because of $m \geq 2$ rank of A is exactly 2), else we may at first slightly change the matrix A without changing signs of its entries and without making its rank greater then 2. All those vectors lie in some two-dimensional plane α. Draw all rays $r_1, \ldots, r_m$ through the origin in directions $v_1, \ldots, v_m$ in α. When we rotate the ray r_1 clockwise until it goes to the position $-r_1$, we meet the rays $\pm r_2, \pm r_3, \ldots, \pm r_m$ in some order. Totally, there are $2^{m-1}(m-1)!$ ways to fix signs and the order of those rays. Note that for any $i = 1, 2, \ldots, n$ the i-th row of A contains the value of some linear functional f_i evaluated in vectors $v_1, \ldots, v_n$. Intersection of the kernel of f_i and α is a line (through the origin, of course), which divides α onto two half-planes, in which f_i takes positive and negative values. Orient this line so that f_i takes positive values on the right hand side of it. The rays $\pm r_i$ divide the plane onto $2m$ angles, and if above oriented line lies in one of those angles, it determines the signs of f_i-values at $v_1, \ldots, v_m$. So, there are at most $(2m)$ possible rows in A'. Totally, we see that there are not more then

$$2^{m-1}(m-1)!(2m)^n < (2m)^{m+n} \leq (m+n)^{m+n}.$$

different sign matrices (we have used obvious inequalities $2^{m-1}(m-1)! \leq (2m)^m$, $2m \leq m+n$).