Problem 1 (Robert Skiba, Nicolaus Copernicus University, Toruń) *Calculate* M^{100}*, where*

$$M = \begin{pmatrix} 1 & 2 & 0 \\ -3 & -3 & 1 \\ 2 & 2 & -1 \end{pmatrix}.$$

Solution 1 *First, we will show that there exist* $a, b, c \in \mathbb{R}$ *such that*

$$M^{100} = aM^2 + bM + cI_3, \tag{1}$$

where I_3 *is the identity matrix. The characteristic polynomial of* M *is* $\chi_M(t) = \det(tI_3 - M) = (t+1)^3$. *In addition, by the Euclidean Algorithm, there is a polynomial* $w(t)$ *and constants* $a, b, c \in \mathbb{R}$ *such that*

$$t^{100} = w(t)\chi_M(t) + at^2 + bt + c. \tag{2}$$

Differentiating twice this equation with respect to t*, we get*

$$100t^{99} = \dot{w}(t)\chi_M(t) + w(t)\dot{\chi}_M(t) + 2at + b, \tag{3}$$

$$9900t^{98} = \ddot{w}(t)\chi_M(t) + 2\dot{w}(t)\dot{\chi}_M(t) + w(t)\ddot{\chi}_M(t) + 2a. \tag{4}$$

Consequently, substituting -1 *into* $(2) - (4)$*, we obtain the following system:*

$$\begin{cases} 1 = a - b + c \\ -100 = -2a + b \\ 9900 = 2a. \end{cases}$$

Hence, $a = 4950,\ b = 9800$ *and* $c = 4851$. *Now, substituting* M *into* (2) *and using the Cayley–Hamilton theorem (which states that every square matrix satisfies its own characteristic polynomial), we deduce that*

$$M^{100} = 4950M^2 + 9800M + 4851I. \tag{5}$$

It is not hard to calculate that

$$M^2 = \begin{pmatrix} -5 & -4 & 2 \\ 8 & 5 & -4 \\ -6 & -4 & 3 \end{pmatrix}. \tag{6}$$

Finally, taking into account (5) *and* (6)*, we get*

$$M^{100} = 4950 \begin{pmatrix} -5 & -4 & 2 \\ 8 & 5 & -4 \\ -6 & -4 & 3 \end{pmatrix} + 9800 \begin{pmatrix} 1 & 2 & 0 \\ -3 & -3 & 1 \\ 2 & 2 & -1 \end{pmatrix} + 4851 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -10099 & -200 & 9900 \\ 10200 & 201 & -10000 \\ -10100 & -200 & 9901 \end{pmatrix}.$$

This completes the solution.