Problem №1

Prove inequality $\int_0^{\pi/2} \frac{x}{\sin x} dx \le \frac{\pi^3}{16}$

Proof:

Lemma **(Chebyshev):** If $f(x)$ and $g(x)$ have different character of monotonicity on $[a;b]$, and $p(x)$ - a certain non-negative on $[a;b]$ function, $p(x)$, $p(x)f(x)$ and $p(x)g(x)$ are integrable on $[a;b]$, then

$$\int_a^b p(x)dx \cdot \int_a^b p(x)f(x)g(x)dx \le \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx .$$

Proof of the lemma:

Consider $\Delta = \int_a^b p(x)f(x)g(x)dx \int_a^b p(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx =$

$$= \int_a^b p(x)f(x)g(x)dx \int_a^b p(y)dy - \int_a^b p(x)f(x)dx \int_a^b p(y)g(y)dy =$$

$$= \int_a^b \int_a^b p(x)p(y)f(x)(g(x)-g(y))dxdy = \int_a^b \int_a^b p(y)p(x)f(y)(g(y)-g(x))dydx$$

Therefore, $\Delta = \frac{1}{2}\int_a^b \int_a^b p(y)p(x)(f(x)-f(y))(g(x)-g(y))dydx$

Due to different monotonicity of the functions $f(x)$ and $g(x)$, differences $(f(x)-f(y))$ and $(g(x)-g(y))$ have different signs, that means that their product is always non-positive, and functions $p(x)$ and $p(y)$ non-negative (the lemma condition), so we integrate non-positive function => $\Delta \le 0$. The lemma was proved.

$\int_0^{\pi/2} \frac{x}{\sin x} dx = \int_0^{\pi/2} \frac{x}{\sin x} \sin x \frac{1}{\sin x} dx$. Let $f(x) = \frac{x}{\sin x}$, $g(x) = \frac{1}{\sin x}$, $p(x) = \sin x$. It is obvious, that function $p(x)$ on the entire interval is non-negative and integrable, $g(x)$ decreases. Show that $f(x)$ increases: $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} = \frac{\cos x(tgx - x)}{\sin^2 x} \ge 0$ (here we used well-known inequality $tgx > x, 0 < x < \frac{\pi}{2}$).

Hence, $\int_0^{\pi/2} \frac{x}{\sin x} dx \le \frac{\int_0^{\pi/2} \frac{x}{\sin x} \sin x dx \int_0^{\pi/2} \frac{1}{\sin x} \sin x dx}{\int_0^{\pi/2} \sin x dx} = \frac{\int_0^{\pi/2} x dx \int_0^{\pi/2} dx}{\int_0^{\pi/2} \sin x dx} = \frac{\pi^3}{16}$.