Problem 5. Given positive integer n. The polynomial $f(x)$ of degree $2n-1$ is so that $f - f^2$ is divisible by $x^n(1-x)^n$. Find all possible values of the leading coefficient of f.

Solution. We have $f - f^2 = f(1-f)$, multiples are coprime, so one of them is divisible by x^n and other by $(1-x)^n$ (if f or $1-f$ multiple is divisible by both x^n and $(1-x)^n$, degree of f may not be equal to $2n-1$). Without loss of generality, x^n divides f and $(1-x)^n$ divides $1-f$, else replace f to $1-f$ (but take in mind that the leading coefficient changes sign.) Now such polynomial f is (at most) unique, as if g is another, there difference $f-g$ is divisible by $x^n(1-x)^n$ and so must vanish. Now we provide an explicit formula for f:

$$f(x) = \frac{\int_0^x t^{n-1}(1-t)^{n-1}dt}{\int_0^1 t^{n-1}(1-t)^{n-1}dt}.$$

Indeed, $f(x)$ is clearly divisible by x^n, and relation $f(1-x) = 1-f(x)$ shows that $f-1$ is divisible by $(1-x)^n$. It remains to calculate the leading coefficient of f. It equals $\frac{(-1)^{n-1}}{(2n-1)I}$, where $I = \int_0^1 t^{n-1}(1-t)^{n-1}dt = B(n,n) = ((n-1)!)^2/(2n-1)!$. So, the final answer is $\pm\binom{2n-2}{n-1}$.