Problem 1 (Robert Skiba, Nicolaus Copernicus University, Toruń) *Let $x(t)$ be a nontrivial solution to the system*

$$\frac{dx}{dt} = Ax, \tag{1}$$

where

$$A = \begin{pmatrix} 1 & 6 & 2 \\ -4 & 4 & 7 \\ -2 & -3 & 7 \end{pmatrix}.$$

Prove that $\mathbb{R} \ni t \mapsto ||x(t)|| \in \mathbb{R}$ is an increasing function, where $||\cdot||$ denotes the Euclidean norm.

Solution 1 *Since $x(t)$ is a nontrivial solution to (1), it follows that $x(t) \neq 0$ for all $t \in \mathbb{R}$. Now, let us observe that it suffices to show that*

$$\frac{d}{dt}||x(t)||^2 > 0$$

for all $t \in \mathbb{R}$. We have

$$\begin{aligned} \frac{d}{dt}||x(t)||^2 = \frac{d}{dt}\langle x(t), x(t)\rangle &= \langle \dot{x}(t), x(t)\rangle + \langle x(t), \dot{x}(t)\rangle \\ &= \langle Ax(t), x(t)\rangle + \langle x(t), Ax(t)\rangle \\ &= \langle Ax(t), x(t)\rangle + \langle A^T x(t), x(t)\rangle \\ &= \langle (A + A^T)x(t), x(t)\rangle. \end{aligned}$$

On the other hand,

$$A + A^T = \begin{pmatrix} 1 & 6 & 2 \\ -4 & 4 & 7 \\ -2 & -3 & 7 \end{pmatrix} + \begin{pmatrix} 1 & -4 & -2 \\ 6 & 4 & -3 \\ 2 & 7 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 8 & 4 \\ 0 & 4 & 14 \end{pmatrix}.$$

Since the determinant of the principal minors are positive:

$$\det(2) > 0, \quad \det\begin{pmatrix} 2 & 2 \\ 2 & 8 \end{pmatrix} > 0 \quad \textit{and} \quad \det\begin{pmatrix} 2 & 2 & 0 \\ 2 & 8 & 4 \\ 0 & 4 & 14 \end{pmatrix} > 0,$$

we deduce that $A + A^T$ is positive definite, which implies that

$$\langle (A + A^T)x(t), x(t)\rangle > 0$$

for all $t \in \mathbb{R}$. Consequently, we have proved that

$$\frac{d}{dt}||x(t)||^2 > 0$$

for all $t \in \mathbb{R}$. This completes the solution.