Problem 1. Are there continuously differentiable functions f, g such that they are not constant on some interval and the following relation takes place at each point of this interval:

$$(f(x)g(x))' = f'(x)g'(x)?$$

Answer Yes. Examples: $f(x) = (x-1)e^x$, $g(x) = e^{\frac{x^2}{2}}$, $x \in (-\infty, +\infty)$; $f(x) = g(x) = \exp(2x)$; etc. .

Solution.Let us make a replacement $f(x) = e^{u(x)}$, $g(x) = e^{v(x)}$. Then, $u'(x) + v'(x) = u'(x)v'(x)$ or $u'(x) - 1 = \frac{1}{v'(x)-1}$. One can choose functions satisfying this condition, e.g., $v(x) = x^2/2$, $u(x) = x + \ln(x-1)$, $x > 1$; $u(x) = v(x) = 2x$ or others. It gives one examples: $f(x) = (x-1)e^x$, $g(x) = e^{\frac{x^2}{2}}$, $x \in (-\infty, +\infty)$; $f(x) = g(x) = \exp(2x)$.

Problem 2. Let $\omega_1, \ldots, \omega_{2016}$ be different roots of degree 2016 of 1. Calculate $\prod_{k \neq l}(\omega_k - \omega_l)$.

Answer: -2016^{2016}.

Solution. Mark $n = 2016$. Fix any k and put $f_k(x) = \prod_{l \neq k}(x - \omega_l)$. Then $f_k(x) = \prod_l (x - \omega_l)/(x - \omega_k) = (x^n - 1)/(x - \omega_k) = (x^n - \omega_k^n)/(x - \omega_k) = x^{n-1} + x^{n-2}\omega_k + x^{n-3}\omega_k^2 + \ldots + \omega_k^{n-1}$. It implies that $f_k(\omega_k) = \omega_k^{n-1} + \omega_k^{n-1} + \ldots \omega_k^{n-1} = n\omega_k^{n-1} = n/\omega_k$. Thus $\prod_{k \neq l}(\omega_k - \omega_l) = \prod_{k=1}^n f_k(\omega_k) = \prod_{k=1}^n (n/\omega_k) = -(-1)^n n^n$.

Problem 3. Prove that

$$I = \int_0^\infty ... \int_0^\infty \frac{dx_1...dx_n}{1 + x_1^{p_1} + ... + x_n^{p_n}} > 1$$

for any positive numbers $p_1...p_n$ such that the integral converges.

Solution. Denote $\Omega_0 = (0,1)^n, \Omega_i = (0,1)^{i-1} \times (1,\infty) \times (0,1)^{n-i-1}$ ($i-$th multiple is $(1,\infty)$, others are $(0,1)$). When integrating over Ω_i , we make a change of variable $x_i \to x_i^{-1}$ which reduces the integral to

$$\int_{\Omega_i} ... = \int_{\Omega_0} \frac{x_i^{-2} dx_1...dx_n}{1 + x_1^{p_1} + ... + x_i^{-p_i} + ... + x_n^{p_n}}.$$

Clearly, for $0 < x_i < 1$, we have

$$\frac{x_i^{-2}}{1 + x_1^{p_1} + ... + x_i^{-p_i} + ... + x_n^{p_n}} > \frac{1}{1 + x_1^{p_1} + ... + x_i^{-p_i} + ... + x_n^{p_n}} >$$

$$\frac{x_i^{p_i}}{1 + x_1^{p_1} + ... + x_i^{p_i} + ... + x_n^{p_n}}.$$

Note that $I > \sum_{i=0}^n \int_{\Omega_i} ...$. Summarizing the above inequalities, one obtains

$$I > \sum_{i=0}^n \int_{\Omega_i} ... >$$

$$\int_{\Omega_0} \frac{dx_1...dx_n}{1 + x_1^{p_1} + ... + x_i^{-p_i} + ... + x_n^{p_n}} + \sum_{i=1}^n \int_{\Omega_0} \frac{x_i^{p_i} dx_1...dx_n}{1 + x_1^{p_1} + ... + x_i^{-p_i} + ... + x_n^{p_n}} = 1.$$

Problem 4. Calculate the exact value of the integral

$$\int_0^{+\infty} 2^{-x} \frac{2^{x-1} - 1 + 2^{-x-1}}{x^2} dx.$$

Answer: $(\ln 2)^2$.

Solution.

We will use the following theorem

Theorem (Frullani's integral). Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous, real valued function such that there exist limits $f_0 = \lim_{x\to 0+} f(x)$, $f_\infty = \lim_{x\to+\infty} f(x)$. Then for every $\alpha, \beta \in \mathbb{R}_+$ the following holds

$$\int_0^{+\infty} \frac{f(\alpha x) - f(\beta x)}{x} dx = (f_\infty - f_0) \ln \frac{\alpha}{\beta}.$$

Proof in the case of $f \in C^2(\mathbb{R}_+)$.

$$\int_0^{+\infty} \frac{f(\alpha x) - f(\beta x)}{x} dx = \int_0^{+\infty} \int_\beta^\alpha f'(xt) dt dx = \int_\beta^\alpha \int_0^{+\infty} f'(xt) dt dx =$$

$$\int_\beta^\alpha (f_\infty - f_0) \frac{dt}{t} (f_\infty - f_0) \ln \frac{\alpha}{\beta}.$$

Solution. We have

$$2^{-x} \frac{2^{x-1} - 1 + 2^{-x-1}}{x^2} = \frac{2^{-1} - 2^{-x} + 2^{-2x-1}}{x^2} = \frac{\frac{1-2^{-x}}{x} - \frac{1-2^{-2x}}{2x}}{x}.$$

Function $f(x) = \frac{1-2^{-x}}{x}$ satisfies the assumptions of the theorem with $f_0 = \ln 2$, $f_\infty = 0$. The integral in question is Frullani's integral with $\alpha = 1, \beta = 2$. It gives the answer: $-\ln 2 \ln \frac{1}{2} = (\ln 2)^2$

Problem 5. Problem 5. Let f be smooth $2\pi-$periodic function. For any segment I of length $|I|$, we determine $f_I = \frac{1}{|I|}\int_I f(t)dt$. For any f we introduce $\|f\| = \sup_I \frac{1}{|I|}\int_I |f(t) - f_I|dt$. Let I and J, $I \subset J$, be segments with the common middle point. Prove that

$$|f_I - f_J| \leq 2(\log_2 \frac{|J|}{|I|} + 1)\|f\|.$$

Solution.

1) At first, consider the case $|J| \leq 2|I|$. It will be the induction base. One has

$$|f_I) - f_J| = \frac{1}{|I|}|\int_I f(t) - f_J dt| \leq \frac{1}{|I|}\int_I |f(t) - f_J|dt \leq$$
$$\frac{2}{|J|}\int_J |f(t) - f_J|dt \leq 2\|f\|.$$

this proves our inequality for the first case.

2) Let us assume that the inequality is valid for any segment $|J'|$ (having the common middle point with I) such that $|J'| \leq 2^n|I|$. Prove the inequality for segments J such that $2^n|I| < |J| \leq 2^{n+1}|I|$. Let us choose the segment J', $|J'| = |J|/2$, $|J'| \leq 2^n|I|, n > 1$, with the same middle point as I, J. Correspondingly, $I \subset J" \subset J$. Due to the induction assumption,

$$|f_I - f_{J'}| \leq 2(\log_2 \frac{|J'|}{|I|} + 1)\|f\|.$$

As $|J| = 2|J'|$, the first part of the proof gives one $|f_J - f_{J'}| \leq 2\|f\|$. Correspondingly, for $2^n|I| < |J| \leq 2^{n+1}|I|$, one has

$$|f_J - f_I| \leq |f_J - f_{J'}| + |f_{J'} - f_I| \leq$$

$$2\|f\| + 2\|f\|(\log_2 \frac{|J'|}{|I|} + 1) \leq 2\|f\|(\log_2 \frac{|J|}{|I|} + 1).$$

The proof is complete.

Remark. There are additional assumptions in the problem condition which are not used.

Problem 6. Let A be a positive definite symmetric real $n \times n$ matrix. Assume that all entries of A are non-negative. Let c_i denote the sum of entries in i-th row of A, $i = 1, \dots, n$. Let k be the sum of all n^2 entries of the matrix A^{-1}. Prove that $k \geqslant 1/c_1 + 1/c_2 + \cdots + 1/c_n$.

Solution. Define positive numbers $m_i = c_i^{-1}/(1/c_1 + \cdots + 1/c_n)$. Then $\sum m_i = 1$, $\sum c_i^{-1} = (\sum m_i^2 c_i)^{-1}$, so we have to prove $k(\sum m_i^2 c_i) \geqslant 1$. Let e_i denote i-th basic unit vector, $v = \sum_{i=1}^n e_i$ be a vector with all coordinates equal to 1. Also denote $B = A^{1/2}$. We have $c_i = (Ae_i, v) = (Be_i, Bv)$, where (x, y) denotes a standard inner product in $\mathbb{R}^n$. Thus

$$\sum_i m_i^2 c_i = \sum_i m_i^2 \sum_j (Be_i, Be_j) \geqslant \sum_i \sum_j m_i m_j (Be_i, Be_j) = \|B\left(\sum m_i e_i\right)\|^2.$$

Next,

$$k = \sum_i \sum_j (B^{-2} e_i, e_j) = \sum_i \sum_j (B^{-1} e_i, B^{-1} e_j) = \|B^{-1} v\|^2.$$

So,

$$k(\sum m_i^2 c_i) = \|B^{-1} v\|^2 \cdot \|B(\sum m_i e_i)\|^2 \geqslant (B^{-1} v, B(\sum m_i e_i))^2 =$$

$$(v, \sum m_i e_i)^2 = (\sum m_i)^2 = 1$$

as desired.

Problem 7. For vectors $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ in $\mathbb{C}^n$ we denote $\langle x, y\rangle = \sum_{i=1}^n x_i \bar{y}_i$. Prove that if $\langle x, x\rangle = \langle y, y\rangle = \langle z, z\rangle = 1$ then

$$\Re\left((1 - \langle x, y\rangle)(1 - \langle y, z\rangle)(1 - \langle z, x\rangle)\right) \geq 0.$$

(here $\Re a$ denotes the real part of a complex number a.)

Solution. Denote $A = \langle x, y\rangle$, $B = \langle z, x\rangle$, $C = \langle z, x\rangle$. The following matrix is non-negative definite as the Gram matrix of x, y, z:

$$M = \begin{pmatrix} 1 & A & \bar{B} \\ \bar{A} & 1 & C \\ B & \bar{C} & 1 \end{pmatrix}, \det M > 0.$$

We have

$$\det M = 1 - |A|^2 - |B|^2 - |C|^2 + 2\Re(ABC) = (1 - |A|^2)(1 - |B|^2) - |C - \bar{A}\cdot\bar{B}|^2 \geq 0.$$

Thus $C = \bar{A}\cdot\bar{B} + w,\ |w| \leq R := \sqrt{(1 - |A|^2)(1 - |B|^2)}$. Under these conditions the best lower estimate for $\Re(1 - A)(1 - B)(1 - C)$ is

$$\Re(1 - A)(1 - B)(1 - C) \geq \Re(1 - A)(1 - B)(1 - \bar{A}\cdot\bar{B}) - R\cdot|1 - A|\cdot|1 - B|.$$

Now we have

$$\begin{aligned} X := \Re(1 - A)(1 - B)(1 - \bar{A}\cdot\bar{B}) = \Re(1 - A)(1 - B)\left((1 - \bar{A}) + \bar{A}\cdot(1 - \bar{B})\right) = \\ = |1 - A|^2(1 - \Re B) + |1 - B|^2(\Re A - |A|^2) \end{aligned}$$

Analogously $X = |1 - A|^2(\Re B - |B|^2) + |1 - B|^2(1 - \Re A)$. Taking half sum of two expressions for X and applying AM-GM for two summands we get

$$X = \frac{|1 - A|^2(1 - |B|^2) + |1 - B|^2(1 - |A|^2)}{2} \geq \sqrt{(1 - |B|^2)(1 - |A|^2)}\cdot|1 - A|\cdot|1 - B|,$$

hence $\Re(1 - A)(1 - B)(1 - C) \geq 0$ as desired.