List of Problems. 6-th NCUMC - 2019.
28.04.2019

1. One makes three times extension of the plane XOY in OY direction $((x,y) \to (x,3y))$. Find maximal variation of angle between directional vectors of lines on the plane under this transformation i.e. $\max|\beta-\alpha|$ where α is the angle before the transformation, β is the angle after the transformation.

2. Let

$$a_n = \sum_{j=1}^{n}\sum_{i=1}^{n}\frac{1}{i^2+j^2}$$

for every n. Find

$$\lim_{n\to\infty}\frac{a_n}{\ln(n)}.$$

3. Let $d>1$ be a positive integer, denote $A=\{(x,x^2,\dots,x^d) : 0\leqslant x\leqslant 1\}\subset\mathbb{R}^d$. Let $B=\operatorname{conv}(A)$ be a convex hull of the set A. Denote by v_d the (d-dimensional) volume of B. Prove that there exists constants $c_1,c_2\in(0,1)$ not depending on d such that $c_1^{d^2}<v_d<c_2^{d^2}$ for all $d>1$.

4. Let $\{v_0,\dots,v_{2099}\}\subset\mathbb{R}^{2100}$ be the family of vectors given by the formula

$$\begin{aligned}
v_0 &= (\underbrace{0,\dots,0}_{2019},\underbrace{1,\dots,1}_{81}),\\
v_k &= (\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{2019},\underbrace{1,\dots,1}_{81-k}) \quad \text{for every } 1\leqslant k\leqslant 81,\\
v_{81+k} &= (\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{81},\underbrace{0,\dots,0}_{2019-k}) \quad \text{for every } 1\leqslant k\leqslant 2018.
\end{aligned}$$

Find the dimension of the linear hull of $\{v_0,\dots,v_{2099}\}$.

5. For given integer $n\geqslant 1$ find the least $c>0$ such that that the $n\times n$ matrix $cR^{-1}-D^{-1}$ is non-negative definite for any symmetric positive definite matrix R with diagonal D (in other words, D is obtained from R by replacing all non-diagonal entries to 0).

6. Consider the equation $y''+f(x)y=0$, where $f(x)$ is a monotonically increasing continuous function on $\mathbb{R}$ with $\inf_{x\in\mathbb{R}} f(x)>0$. It is known that any non-trivial solution y to the equation is oscillating, thus having an infinite sequence of zeroes $\{x_i\}$, $y(x_i)=0$, and an infinite sequence of local extrema $\{x_i'\}$, $y'(x_i')=0$, such that $x_i<x_i'<x_{i+1}$. Prove that (i) $|y(x_i')|$ decreases, (ii) $|y'(x_i)|$ increases.

7. Let f be an analytic function in $D=\{z:|z|<1\}$ such that $|f(z)|\le 1$. Prove that for $z\in D$ one has

$$\frac{|f'(z)|}{1-|f(z)|^2}\le\frac{1}{1-|z|^2}.$$

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7. Let f be an analytic function in $D = \{z : |z| < 1\}$ such that $|f(z)| \le 1$. Prove that for $z \in D$ one has

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$