List of Problems. 6-th NCUMC - 2019.
28.04.2019

1. One makes three times extension of the plane $XOY$ in $OY$ direction $((x, y) \to (x, 3y))$. Find maximal variation of angle between directional vectors of lines on the plane under this transformation i.e. $\max|\beta - \alpha|$ where $\alpha$ is the angle before the transformation, $\beta$ is the angle after the transformation.

2. Let
$$a_n = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{i^2 + j^2}$$
for every $n$. Find
$$\lim_{n\to\infty} \tfrac{a_n}{\ln(n)}.$$

3. Let $d > 1$ be a positive integer, denote $A = \{(x, x^2, \ldots, x^d) : 0 \leqslant x \leqslant 1\} \subset \mathbb{R}^d$. Let $B = \operatorname{conv}(A)$ be a convex hull of the set $A$. Denote by $v_d$ the ($d$-dimensional) volume of $B$. Prove that there exists constants $c_1, c_2 \in (0, 1)$ not depending on $d$ such that $c_1^{d^2} < v_d < c_2^{d^2}$ for all $d > 1$.

4. Let $\{v_0, \ldots, v_{2099}\} \subset \mathbb{R}^{2100}$ be the family of vectors given by the formula
$$\begin{aligned} v_0 &= (\underbrace{0,\ldots,0}_{2019},\underbrace{1,\ldots,1}_{81}), \\ v_k &= (\underbrace{1,\ldots,1}_{k},\underbrace{0,\ldots,0}_{2019},\underbrace{1,\ldots,1}_{81-k}) \quad \text{for every } 1 \leqslant k \leqslant 81, \\ v_{81+k} &= (\underbrace{0,\ldots,0}_{k},\underbrace{1,\ldots,1}_{81},\underbrace{0,\ldots,0}_{2019-k}) \quad \text{for every } 1 \leqslant k \leqslant 2018. \end{aligned}$$
Find the dimension of the linear hull of $\{v_0, \ldots, v_{2099}\}$.

5. For given integer $n \geqslant 1$ find the least $c > 0$ such that that the $n \times n$ matrix $cR^{-1} - D^{-1}$ is non-negative definite for any symmetric positive definite matrix $R$ with diagonal $D$ (in other words, $D$ is obtained from $R$ by replacing all non-diagonal entries to 0).

6. Consider the equation $y'' + f(x)y = 0$, where $f(x)$ is a monotonically increasing continuous function on $\mathbb{R}$ with $\inf_{x\in\mathbb{R}} f(x) > 0$. It is known that any non-trivial solution $y$ to the equation is oscillating, thus having an infinite sequence of zeroes $\{x_i\}$, $y(x_i) = 0$, and an infinite sequence of local extrema $\{x'_i\}$, $y'(x'_i) = 0$, such that $x_i < x'_i < x_{i+1}$. Prove that (i) $|y(x'_i)|$ decreases, (ii) $|y'(x_i)|$ increases.

7. Let $f$ be an analytic function in $D = \{z : |z| < 1\}$ such that $|f(z)| \leq 1$. Prove that for $z \in D$ one has
$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

**Solutions.**

1 (suggested by ITMO University). One makes three times extension of the plane $XOY$ in $OY$ direction ($(x, y) \to (x, 3y)$). Find maximal variation of angle between directional vectors of lines on the plane under this transformation i.e. $\max|\beta - \alpha|$ where $\alpha$ is the angle before the transformation, $\beta$ is the angle after the transformation.

1. Solution. Answer: $\frac{\pi}{3}$.

Line $y = kx + b$ transforms to $y = 3kx + 3b$, i.e. rotates to the angle $\phi = \arctan(3k) - \arctan k$. Let us find extrema of $\phi$.

$$\frac{d\phi}{dk} = \frac{3}{1+9k^2} - \frac{1}{1+k^2} = \frac{2(1-3k^2)}{(1+9k^2)(1+k^2)} = 0.$$

One has maximum for $k = \frac{1}{\sqrt{3}}$, $\max\phi = \frac{\pi}{6}$, and minimum for $k = -\frac{1}{\sqrt{3}}$, $\min\phi = -\frac{\pi}{6}$. Correspondingly, maximal variation of angle is $\max\phi - \min\phi = \frac{\pi}{3}$.

2 (suggested by Adam Mickiewicz University, Poznan). Let

$$a_n = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{i^2 + j^2}$$

for every $n$. Find

$$\lim_{n\to\infty} \tfrac{a_n}{\ln(n)}.$$

2. Solution. Let

$$f(x, y) = \frac{1}{x^2 + y^2}.$$

Since

$$\frac{1}{s^2 + t^2} \leqslant \frac{1}{i^2 + j^2} \leqslant \frac{1}{x^2 + y^2}$$

for every $(s, t) \in [i-1, i] \times [j-1, j]$ and $(x, y) \in [i, i+1] \times [j, j+1]$ and $i, j \in \mathbb{N}$, there holds the inequalities

$$\begin{aligned}
a_n &\leqslant \frac{1}{2} + \sum_{\{(i,j)\in\{1,\dots,n\}^2\setminus\{(1,1)\}\}} \int_{(i-1,i]\times([j-1,j]} \frac{1}{x^2 + y^2}\, dxdy \\
&\leqslant \frac{1}{2} + \int_{\{(x,y)\in\mathbb{R}^2 : x>0, y>0, 1\leqslant x^2+y^2\leqslant 2n^2\}} \frac{1}{x^2 + y^2}\, dxdy \\
&\leqslant \frac{1}{2} + \int_1^{\sqrt{2}n} \int_0^{\frac{\pi}{2}} \frac{r}{r^2}\, d\phi dr = \frac{1}{2} + \frac{\pi}{2} \ln(\sqrt{2}n) = \frac{\pi}{2} \ln(n) + \frac{1}{2} + \frac{\pi}{2} \ln(\sqrt{2})
\end{aligned}$$

and

$$\begin{aligned}
a_n &\geqslant \sum_{\{(i,j)\in\{1,\dots,n\}^2\}} \int_{[i,i+1)\times[j,j+1)} \frac{1}{x^2 + y^2}\, dxdy \\
&\geqslant \int_{[0,n)^2\setminus[0,1)^2} \frac{1}{x^2 + y^2}\, dxdy - \int_{[0,1]\times[1,n)} \frac{1}{x^2 + y^2}\, dxdy - \int_{[1,n)\times[0,1]} \frac{1}{x^2 + y^2}\, dxdy \\
&\geqslant \int_{\{(x,y)\in\mathbb{R}^2 : x>0, y>0, 2\leqslant x^2+y^2<n^2\}} \frac{1}{x^2 + y^2}\, dxdy - 2 \int_1^n \frac{1}{x^2}\, dx \\
&\geqslant \int_{\sqrt{2}}^{n} \int_0^{\frac{\pi}{2}} \frac{r}{r^2}\, d\phi dr - 2(1 - \tfrac{1}{n}) = \frac{\pi}{2}(\ln(n) - \ln(\sqrt{2})) - 2(1 - \tfrac{1}{n}).
\end{aligned}$$

Gather together all the facts above we obtain

$$\lim_{n\to\infty} \tfrac{a_n}{\ln(n)} = \tfrac{\pi}{2}.$$

3 (suggested by Saint Petersburg State University). Let $d > 1$ be a positive integer, denote $A = \{(x, x^2, \dots, x^d) : 0 \leqslant x \leqslant 1\} \subset \mathbb{R}^d$. Let $B = \operatorname{conv}(A)$ be a convex hull of the set $A$. Denote by $v_d$ the ($d$-dimensional) volume of $B$. Prove that there exists constants $c_1, c_2 \in (0,1)$ not depending on $d$ such that $c_1^{d^2} < v_d < c_2^{d^2}$ for all $d > 1$.

3. Solution. Choose the points $p_0, p_1, \dots, p_d$ on the curve $A$ so that the volume $w_d$ of the simplex $T$ with the vertices $p_0, p_1, \dots, p_d$ is maximal. Denote by $\tilde{T}$ the simplex homothetic to $T$ in its barycentre and coefficient $-d$ (in other words, the facets of $\tilde{T}$ are parallel to those of $T$ and pass through respective vertices of $T$.) Then $T \subset A \subset \tilde{T}$ (the second inclusion follows from the maximality of the volume). Therefore $w_d \leqslant v_d \leqslant d^d w_d$ and $\log v_d = \log w_d + o(d^2)$, hence it suffices to prove the same estimate for $w_d$ (this allows to find the constants $c_1, c_2$ working for all large enough $d$, but for bounded $d > 1$ some constants work simply because $0 < v_d < 1$).

If $p_i = (x_i, x_i^2, \dots, x_i^d)$, we have

$$d! w_d = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^d \\ 1 & x_1 & x_1^2 & \dots & x_1^d \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_d & x_d^2 & \dots & x_d^d \end{vmatrix} = \prod_{0 \leqslant i < j \leqslant d} |x_i - x_j|.$$

For estimating this product from above, note that $\sum_{i<j} |x_i - x_j| \leqslant [(d+1)^2/4]$ (for example, we may assume $x_0 < x_1 < \dots < x_d$, then the sum equals $d x_d + (d-2) x_{d-1} + \dots + (-d) x_0 \leqslant d + (d-2) + \dots (d - 2[d/2])) = [(d+1)^2/4]$.) Therefore

$$d! w_d \leqslant \left( \frac{[(d+1)^2/4]}{d(d+1)/2} \right)^{d(d+1)/2} = \left( \frac{1}{2} + o(1) \right)^{d^2/2}.$$

For the lower estimate, note that for $y_i = i/d$ (this is not the best possible, but enough for our goal) we have

$$\prod_{i<j} |y_i - y_j|^2 = \prod_i \frac{i!(d-i)!}{d^d} \geqslant \prod_{i=0}^{d} \left( \frac{1}{2e} \right)^d = \left( \frac{1}{2e} \right)^{d(d+1)}.$$

Here we used the standard inequalities $i! \geqslant (i/e)^i$ (may be proved by induction) and $i^i (d-i)^{d-i} \geqslant (d/2)^d$ (may be proved by taking the derivative in $i$).

Remark. It is known that the maximum of $\prod_{i<j} |x_i - x_j|$ is attained when $x_i$'s are $0, 1$ and the roots of Jacobi polynomial $J_{d-1}(1, 1, x)$. The asymptotics of the maximum is $(2 + o(1))^{-d^2}$.

4 (suggested by Adam Mickiewicz University, Poznan). Let $\{v_0, \ldots, v_{2099}\} \subset \mathbb{R}^{2100}$ be the family of vectors given by the formula

$$v_0 = (\underbrace{0, \ldots, 0}_{2019}, \underbrace{1, \ldots, 1}_{81}),$$
$$v_k = (\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{2019}, \underbrace{1, \ldots, 1}_{81-k}) \quad \text{for every } 1 \leqslant k \leqslant 81,$$
$$v_{81+k} = (\underbrace{0, \ldots, 0}_{k}, \underbrace{1, \ldots, 1}_{81}, \underbrace{0, \ldots, 0}_{2019-k}) \quad \text{for every } 1 \leqslant k \leqslant 2018.$$

Find the dimension of the linear hull of $\{v_0, \ldots, v_{2099}\}$.

4. Solution. We will need the following well known lemma.

**Lemma**

*For every $a_0, \ldots, a_{n-1} \in \mathbb{C}$*

$$\det \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_0 \end{pmatrix} = g(\omega_n^1) \cdot \ldots \cdot g(\omega_n^n)$$

*where*

$$g(x) = \sum_{j=0}^{n-1} a_j x^j$$

*and* $\omega_n = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$.

Let

$$h(x) = \sum_{j=2019}^{2099} x^j = x^{2019} \frac{1 - x^{81}}{1 - x}.$$

For every $\lambda \in \mathbb{C}$

$$\det \left( \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{2099} \end{pmatrix} - \lambda I \right) = (-\lambda + h(\omega_{2100}^1)) \cdot \ldots \cdot (-\lambda + h(\omega_{2100}^{2100})).$$

Hence

$$dim\left(lin\{v_0, \ldots, v_{2099}\}\right) = 2019 + 81 - |\{1 \leqslant k \leqslant 2099 : (\omega_{2100}^k)^{81} = 1\}|.$$

It is easy to see that $3^4 k = 2^2 \cdot 3 \cdot 5^2 \cdot 7p$ for some $p \in \mathbb{N}$ only for $k = 700$ and $k = 1400$. Therefore

$$dim\left(lin\{v_0, \ldots, v_{2099}\}\right) = 2098.$$

*Proof of Lemma*

$$\det \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & \ldots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_0 \end{pmatrix} \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \omega_n & \omega_n^2 & \ldots & \omega_n^n \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{n(n-1)} \end{pmatrix}$$
$$= \det \begin{pmatrix} g(\omega_n) & g(\omega_n^2) & \ldots & g(\omega_n^n) \\ \omega_n g(\omega_n) & \omega_n^2 g(\omega_n^2) & \ldots & \omega_n^n g(\omega_n^n) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^{n-1} g(\omega_n) & \omega_n^{2(n-1)} g(\omega_n^2) & \ldots & \omega_n^{n(n-1)} g(\omega_n^n) \end{pmatrix}$$
$$= g(\omega_n^1) \cdot \ldots \cdot g(\omega_n^n) \cdot \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \omega_n & \omega_n^2 & \ldots & \omega_n^n \\ \vdots & \vdots & \ddots & \vdots \\ \omega_n^{n-1} & \omega_n^{2(n-1)} & \ldots & \omega_n^{n(n-1)} \end{pmatrix}.$$

5 (suggested by Saint Petersburg State University). For given integer $n \geqslant 1$ find the least $c > 0$ such that that the $n \times n$ matrix $cR^{-1} - D^{-1}$ is non-negative definite for any symmetric positive definite matrix $R$ with diagonal $D$ (in other words, $D$ is obtained from $R$ by replacing all non-diagonal entries to 0).

5. Answer: $c = n$.

Let $R$ be a matrix with diagonal entries equal to 1 and off-diagonal entries equal to $1 - t$ for small $t \in (0, 1)$. The eigenvalues of this matrix are $n - nt$ (with eigenvector $u = (1, 1, \ldots, 1)$) and $t$ (with multiplicity $n-1$ and eigenvectors orthogonal to $u$). Thus $R$ is positive definite, and $R^{-1}$ has an eigenvalue $(n - nt)^{-1}$. Therefore $cR^{-1} - D^{-1} = cR^{-1} - I$ has eigenvalue $cn(1 - t) - 1$ and if $c < n$, this is negative for small $t$. Therefore $c \geqslant n$. Now we prove that $c = n$ works. Denote $R = D^{1/2}QD^{1/2}$, then $Q = D^{-1/2}RD^{-1/2}$ is a positive definite symmetric matrix with all diagonal elements equal to 1. And we have to prove that $nR^{-1} - D^{-1} = D^{-1/2}(nQ^{-1} - I)D^{-1/2}$ is non-negative definite. Note that the sum of eigenvalues of $Q$ equals to the trace of $Q$, which equals to $n$. Therefore all eigenvalues of $Q$ belong to $(0, n)$, and all eigenvalues of $Q^{-1}$ belong to $(1/n, \infty)$, that just means that $nQ^{-1} - I$ is positive definite.

6 (suggested by Moscow State University). Consider the equation $y'' + f(x)y = 0$, where $f(x)$ is a monotonically increasing continuous function on $\mathbb{R}$ with $\inf_{x\in\mathbb{R}} f(x) > 0$. It is known that any non-trivial solution $y$ to the equation is oscillating, thus having an infinite sequence of zeroes $\{x_i\}$, $y(x_i) = 0$, and an infinite sequence of local extrema $\{x'_i\}$, $y'(x'_i) = 0$, such that $x_i < x'_i < x_{i+1}$. Prove that (i) $|y(x'_i)|$ decreases, (ii) $|y'(x_i)|$ increases.

6. Solution. Multiplying the equation by $2y'(x)$ and then integrating it over an arbitrary segment $[a; b]$, we obtain

$$y'(b)^2 - y'(a)^2 + \int_a^b 2f(x)\,y(x)\,y'(x)\,dx = 0. \tag{$*$}$$

Put $h_j = y(x'_j)$ and $v_j = y'(x_j)$. Now we use formula (*) for various segments $[a; b]$.

(i) On $(x'_{j-1}; x_j)$ and $(x_j; x'_j)$ we have respectively $y(x)y'(x) < 0$ and $y(x)y'(x) > 0$, whence

$$0 = v_j^2 + \int_{x'_{j-1}}^{x_j} 2f(x)\,y(x)\,y'(x)\,dx > v_j^2 + f(x_j)\int_{x'_{j-1}}^{x_j} 2y(x)\,y'(x)\,dx = v_j^2 - f(x_j)h_{j-1}^2$$

and

$$0 = -v_j^2 + \int_{x_j}^{x'_j} 2f(x)\,y(x)\,y'(x)\,dx > -v_j^2 + f(x_j)\int_{x_j}^{x'_j} 2y(x)\,y'(x)\,dx = -v_j^2 + f(x_j)h_j^2.$$

The sum of the last two inequalities gives $0 > f(x_j)(-h_{j-1}^2 + h_j^2)$, whence $|h_j|$ decreases.

(ii) On $(x_j; x'_j)$ and $(x'_j; x_{j+1})$ we have respectively $y(x)y'(x) > 0$ and $y(x)y'(x) < 0$, whence

$$0 = -v_j^2 + \int_{x_j}^{x'_j} 2f(x)\,y(x)\,y'(x)\,dx < -v_j^2 + f(x'_j)\int_{x_j}^{x'_j} 2y(x)\,y'(x)\,dx = -v_j^2 + f(x'_j)h_j^2$$

and

$$0 = v_{j+1}^2 + \int_{x'_j}^{x_{j+1}} 2f(x)\,y(x)\,y'(x)\,dx < v_{j+1}^2 + f(x'_j)\int_{x'_j}^{x_{j+1}} 2y(x)\,y'(x)\,dx = v_{j+1}^2 - f(x'_j)h_j^2.$$

The sum of the last two inequalities gives $0 < -v_j^2 + v_{j+1}^2$, whence increases.

7 (suggested by ITMO University). Let $f$ be an analytic function in $D = \{z : |z| < 1\}$ such that $|f(z)| \leq 1$. Prove that for $z \in D$ one has

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.$$

7. Solution. Let us prove the following lemma.

**Lemma** (Schwartz lemma). Let $g$ be an analytic function in $D = \{z : |z| < 1\}$ such that $|g(z)| \leq 1$ and $g(0) = 0$ then $|g(z)| \leq |z|, \quad |z| < 1$.

*Proof of the lemma.* Function $h(z) = \frac{g(z)}{z}$ is analytic in $D$. $|h(z)| \leq 1$, $|z| = 1$. In accordance with the maximum principle, $|h(z)| \leq 1$, $|z| < 1$. This proves the lemma.

Let $z, z_0 \in D$ and

$$w(z) = \frac{z - z_0}{1 - \overline{z_0} z}.$$

This is a map of $D$ onto $D$, $z_0 = w^{-1}(0)$. Consider the analytic function

$$\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)}$$

as a function of new variable $w(z)$. Due to the Lemma, one has

$$\left|\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)}\right| \leq \left|\frac{z - z_0}{1 - \overline{z_0} z}\right|, \quad z \neq z_0.$$

Let $z_0 \to z$. Due to the fact that $\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \to |f'(z)|$, one comes to the inequality

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.$$