# Problems for NCUMC 2023
## 23.04.2023

1. Polynomial $P(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+...+1$ with non-negative coefficients has $n$ real roots. Prove that $P(2023)\geq 2024^n$.

2. Function $f$ is twice continuously differentiable on the real axis and satisfies the relation $f''(x)+xg(x)f'(x)+f(x)=0, \forall x,$ where $g(x)\geq 0.$ Prove that $f(x)$ is bounded.

3. Let $A, O_1, O_2, ...O_{2023}$ be 2024 points in $\mathbb{R}^3$. Let point $A_1$ is symmetric to $A$ in respect to $O_1$, point $A_2$ is symmetric to $A_1$ in respect to $O_2$, … point $A_{2023}$ is symmetric to $A_{2022}$ in respect to $O_{2023}$, and, continuing the chain of reflections one obtains: point $A_{2024}$ is symmetric to $A_{2023}$ in respect to $O_1$,… point $A_{4046}$ is symmetric to $A_{4045}$ in respect to $O_{2023}$. Prove that point $|AA_{4046}|\leq \min\limits_{\substack{i,j=1,2,...2023\\ i\neq j}} |O_iO_j|$

4. The surface of a melon has the form $x^4+y^4+z^4=a^4,\ a>0$. Is it possible to cut it by plane in such a way that the cross-section would be a circle?

5. A continuous function $f:[1,2]\to\mathbb{R}$ satisfies the inequalities

$\left|\int_1^2 f(x)x^n dx\right| < 2^{-1000n}$ for every $n=1,2,...$ Prove that $f\equiv 0$.

6. Fyodor the robot prefers a positive integer $n$ if $n^{2022}\geq w^{2023}$, where $w$ is the product of all prime divisors of $n$. Is the sum of reciprocals of numbers preferred by Fyodor the robot finite or not?

7. Consider a regular 2023-gon inscribed in a circle of radius 2023. Let $\Omega$ be the set of all $C^2$ naturally parameterized simple (that is, non-self-intersecting) closed plane curves passing through all 2023 vertices of the polygon. Find the infimum $\inf\limits_{\gamma\in\Omega}\left\{\oint_\gamma (\kappa_\gamma(s))^2 ds\right\}$, where $\kappa_\gamma(s)$ is the curvature of $\gamma$ at the point with parameter s.

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1. Polynomial $P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + 1$ with non-negative coefficients has $n$ real roots. Prove that $P(2023) \geq 2024^n$.

1. Solution. Coefficients of the polynomial are non-negative, the last coefficient equals one. It means that all roots $x_k$ are negative. Пусть его корни $x_k = -b_k, b_k > 0, k = 1, 2, ...n$. Then $P\left(x\right) = \left(x + b_1\right)\left(x + b_2\right)...\left(x + b_n\right)$. The inequality between means gives one: $2023 + b_k = 1 + 1 + ... + 1 + b_k \geq 2024 \sqrt[2024]{b_k}$. Correspondingly, $P\left(2023\right) \geq 2024^n \sqrt[2024]{b_1 \cdot b_2 \cdot ... \cdot b_n} = 2024^n$ due to the Viete theorem $(b_1 \cdot b_2 \cdot ... \cdot b_n = 1)$.

2. Function $f$ is twice continuously differentiable on the real axis and satisfies the relation $f''(x)+xg(x)f'(x)+f(x)=0, \forall x,$ where $g(x)\geq 0$. Prove that $f(x)$ is bounded.

2. Let us multiply the equation by $2f'(x)$ and integrate it over $[0,x]$. One obtains $\left(f'(x)\right)^2-\left(f'(0)\right)^2+\left(f(x)\right)^2-\left(f(0)\right)^2=-2\int_0^x t\cdot g(t)\cdot\left(f'(t)\right)^2 dt$.

The right hand side of the equality is non-positive. Hence, $\left(f(x)\right)^2+\left(f'(x)\right)^2-\left(f(0)\right)^2-\left(f'(0)\right)^2\leq 0$. As a result, one obtains the estimation $\left(f(x)\right)^2\leq\left(f(x)\right)^2+\left(f'(x)\right)^2\leq\left(f(0)\right)^2+\left(f'(0)\right)^2$, which means that $f(x)$ is bounded.

3. Let $A, O_1, O_2, ...O_{2023}$ be 2024 points in $\mathbb{R}^3$. Let point $A_1$ is symmetric to $A$ in respect to $O_1$, point $A_2$ is symmetric to $A_1$ in respect to $O_2$, … point $A_{2023}$ is symmetric to $A_{2022}$ in respect to $O_{2023}$, and, continuing the chain of reflections one obtains: point $A_{2024}$ is symmetric to $A_{2023}$ in respect to $O_1$,… point $A_{4046}$ is symmetric to $A_{4045}$ in respect to $O_{2023}$. Prove that point $|AA_{4046}| \le \min_{\substack{i,j=1,2,...2023 \\ i \ne j}} |O_i O_j|$

3. Solution. Two consequent reflections in respect to points $O_1$ and $O_2$ gives one a shift to vector $2\overrightarrow{O_1O_2}$ due to the fact that $O_1O_2$ is the midline of a triangle $\Delta AA_1A_2$. Consequently, $\overrightarrow{AA_2} = 2\overrightarrow{O_1O_2}$. If $A$, $A_1$ and $A_2$ belong to one straight line the result is, evidently, the same. Thus, one has $\overrightarrow{AA_2} = 2\overrightarrow{O_1O_2}$, $\overrightarrow{A_2A_4} = 2\overrightarrow{O_3O_4}$, … $\overrightarrow{A_{2020}A_{2022}} = 2\overrightarrow{O_{2021}O_{2022}}$, $\overrightarrow{A_{2022}A_{2024}} = 2\overrightarrow{O_{2023}O_1}$, $\overrightarrow{A_{2024}A_{2026}} = 2\overrightarrow{O_2O_3}$,… $\overrightarrow{A_{4044}A_{4046}} = 2\overrightarrow{O_{2022}O_{2023}}$. Let us find

$$\overrightarrow{AA_{4046}} = \overrightarrow{AA_2} + \overrightarrow{A_2A_4} + ... + \overrightarrow{A_{4044}A_{4046}} =$$

$$= 2\overrightarrow{O_1O_2} + 2\overrightarrow{O_3O_4} + ... + 2\overrightarrow{O_{2021}O_{2022}} + 2\overrightarrow{O_{2023}O_1} + 2\overrightarrow{O_2O_3} + ... + 2\overrightarrow{O_{2022}O_{2023}} =$$

$$= 2(\overrightarrow{O_1O_2} + \overrightarrow{O_2O_3} + \overrightarrow{O_2O_3} + ... + 2\overrightarrow{O_{2022}O_{2023}} + 2\overrightarrow{O_{2023}O_1}) = \vec{0}.$$

Thus, $|AA_{4046}| = 0$.

4. The surface of a melon has the form $x^4 + y^4 + z^4 = a^4$, $a > 0$. Is it possible to cut it by plane in such a way that the cross-section would be a circle?

4. Answer. Yes. Consider the following circle: $\begin{cases} x + y + z = 0 \\ x^2 + y^2 + z^2 = r^2 \end{cases}$. One has for the points of this circle $x + y = -z \Rightarrow x^2 + y^2 + 2xy = z^2, 2xy = z^2 - x^2 - y^2 = 2z^2 - r^2$. Hence, $x^4 + y^4 + z^4 = (x^2 + y^2)^2 - 2x^2y^2 + z^4 = (r^2 - z^2)^2 - \frac{1}{2}(2z^2 - r^2)^2 + z^4 = \frac{r^4}{2}$. It means that the following circle $\begin{cases} x + y + z = 0, \\ x^2 + y^2 + z^2 = \sqrt{2}a^2 \end{cases}$ (with the radius $\sqrt[4]{2}a$ and the center at the origin) belongs to the surface $x^4 + y^4 + z^4 = a^4$, $a > 0$.

5. A continuous function $f\colon [1,2]\to\mathbb{R}$ satisfies the inequalities $|\int_1^2 f(x)x^n dx| < 2^{-1000n}$ for every $n=1,2,\ldots$. Prove that $f\equiv 0$.

Solution

5. Put $\epsilon = 1/100000$, then $\int_0^1 f(x)x^n = O(\epsilon^n)$.

**Lemma**. There exists a polynomial $p_n(x)=\sum_{k=n}^{10n} c_k x^k$ such that $\sum_{k=n}^{10n}|c_k|\epsilon^k = o(1)$ and $\max_{x\in[1,2]}|1-p_n(x)| = o(1)$.

First of all, I prove the claim using the polynomials from lemma. It suffices to prove that $\int_1^2 f(x)x^d dx = 0$ for arbitrary non-negative integer $d$ (then $f$ is orthogonal to all polynomials, and by Weierstrass theorem is identical 0). We have

$$\int_1^2 f(x)x^d dx = \int_1^2 f(x)x^d(1-p_n(x))dx + \int_1^2 f(x)x^d p_n(x)dx = o(1) + \sum_{k=n}^{10n} c_k \int_1^2 f(x)x^{d+k}dx = o(1),$$

and the claim follows.

Proof of the lemma. Consider the Taylor approximation of degree $9n$ at point $3/2$ of the function $F(x) := x^{-n}$:

$$x^{-n} = \sum_{k=0}^{9n} \frac{n(n+1)\cdots(n+k-1)}{k!}(3/2)^{-n-k}(x-3/2)^k + R_{9n}(x),$$

where the remainder term $R_{9n}(x) = \frac{F^{(9n+1)(\theta)}}{(9n+1)!}(x-3/2)^{9n+1}$ for an intermediate point $\theta$ between $3/2$ and $x$ enjoys on $[1,2]$ the bound

$$|R_{9n}(x)| \leqslant (1/2)^{9n+1}\cdot\frac{n(n+1)\cdots(n+9n)}{(9n+1)!} = 2^{-9n-1}\binom{10n}{n-1} < 2^{-9n-1}\frac{(10n)^n}{n!} < (10e\cdot 2^{-9})^n < 3^{-n},$$

thus $x^n R_{9n}(x)$ is uniformly small on $[1,2]$. Put

$$p_n(x) = x^n \sum_{k=0}^{9n} \frac{n(n+1)\cdots(n+k-1)}{k!}(3/2)^{-n-k}(x-3/2)^k = \sum_{k=n}^{10n} c_n x^n.$$

We have

$$\sum_{k=n}^{10n}|c_n|\epsilon^n \leqslant \epsilon^n \sum_{k=0}^{9n}\frac{n(n+1)\cdots(n+k-1)}{k!}(3/2)^{-n-k}(\epsilon+3/2)^k$$

$$< \epsilon^n(3/2)^{-n}\sum_{k=0}^{9n}\binom{n+k-1}{k}\cdot 2^k < \epsilon^n(3/2)^{-n}\sum_{k=0}^{9n}(1+2)^{n+k-1} = o(1)$$

as needed.

2. Fyodor the robot prefers a positive integer $n$ if $n^{2022} \geqslant w^{2023}$, where $w$ is the product of all prime divisors of $n$. Is the sum of reciprocals of numbers preferred by Fyodor the robot finite or not?

Solution

2. The sum converges. Fix $w = p_1 \dots p_k$, where $p_i$ are all distinct prime divisors of $n$. Denote by $\Omega$ the set of positive integers with all prime divisors in $\{p_1, \dots, p_k\}$. Then $n = wQ$ where $Q \in \Omega$ and $Q \geqslant w^{1/2023}$. For $s \in (0,1)$ we have

$$\prod_{i=1}^{k}(1-p^{s-1})^{-1} = \prod_{i=1}^{k}(1+p^{s-1}+p^{2(s-1)}+\dots) = \sum_{Q\in\Omega} Q^{s-1} \geqslant w^{s/2023} \sum_{Q\in\Omega, Q\geqslant w^{1/2023}} Q^{-1}.$$

Thus

$$\sum_{Q\in\Omega, Q\geqslant w^{1/2023}} Q^{-1} \leqslant \prod_{i=1}^{k} p_i^{-s/2023}(1-p_i^{s-1})^{-1} = \prod_{i=1}^{k}(p_i^{s/2023} - p_i^{s/2023+s-1})^{-1}.$$

From now we fix $s = 1/2$. We get

$$\sum_{n \text{ preferred by FtR}} n^{-1} \leqslant$$

$$\sum_{p_1,\dots,p_k} \prod_{i=1}^{k} \frac{1}{p_i(p_i^{1/4046} - p_i^{1/4046-1/2})} = \prod_{p}\left(1 + \frac{1}{p(p^{1/4046} - p^{1/4046-1/2})}\right) < \infty,$$

since $1+x < e^x$ for $x = (p(p^{1/4046} - p^{1/4046-1/2}))^{-1}$, and the sum of $(p(p^{1/4046} - p^{1/4046-1/2}))^{-1}$ is finite even taken over all integers $p > 1$, not necessarily prime.

**Problem 7.**

Consider a regular 2023-gon inscribed in a circle of radius 2023. Let $\Omega$ be the set of all $C^2$ naturally parametrized simple (that is, non-self-intersecting) closed plane curves passing through all 2023 vertices of the polygon. Find the infimum

$$\inf\left\{\oint_\gamma \kappa_\gamma(s)^2\,ds : \gamma\in\Omega\right\},$$

where $\kappa_\gamma(s)$ is the curvature of $\gamma$ at the point with parameter $s$.

**Answer:** 0.

**Solution.**

We are going to construct a curve $\gamma\in\Omega$ having an arbitrarily small value of the above integral. The curve will consist of line segments (with the above integral equal to zero) and special curve arcs defined by $\kappa_\gamma(s)=\lambda s(D-s),\ \ s\in[0;D]$, where the constant $\lambda$ is chosen so that the angle between the tangent vectors at the end points is equal to $\pi/2$ :

$$\int_0^D \kappa_\gamma(s)ds=\lambda\int_0^D s(D-s)ds=\frac{\lambda D^3}{6}=\pm\frac{\pi}{2},\quad\text{whence}\quad \lambda=\pm 3\pi D^{-3}.$$

If the special arc has a horizontal or vertical tangent vector at one of its end-points, then the horizontal and vertical changes between end-points are equal and are denoted by $W_D$ or just $W$.

Note that

$$\int_0^D \kappa_\gamma(s)^2ds=\lambda^2\int_0^D s^2(D-s)^2ds=\frac{\lambda^2D^5}{30}=0.3\pi^2D^{-1}.$$

So, the greater is $D$, the longer is the arc, but the smaller is the integral of its curvature squared.

Since such arcs have zero curvature at their end-points, they can be joined to form a $C^2$ curve. The same holds when joining such an arc with a line segment.

We enumerate the vertices as $p_1,\dots,p_{2023}$ when passing them clockwise along the circumscribed circle and suppose that the edge $[p_1,p_2]$ is horizontal and above all other vertices. The curve to be constructed will pass the vertices in the following order:

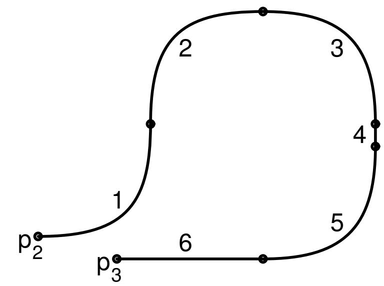
$$p_1,p_2,p_3,p_{2023},p_{2022},p_4,p_5,p_{2021},\dots,p_{1014},p_{1012},p_{1013},p_1.$$

The first part of the curve, from $p_1$ to $p_2$, is a horizontal line segment, as well as all parts from $p_{2k+1}$ to $p_{2025-2k}$ and from $p_{2024-2k}$ to $p_{2k+2}$, $k=1,\dots,505$.

Now we describe the second part, from $p_2$ to $p_3$, supposing $W>R=2023$ (see the figure below).

This part consists successively of:

1,2,3) three special arcs turning up, right and down,




4) a vertical line segment of length equal to the vertical change between $p_2$ and $p_3$,

5) a special arc turning left,

6) a horizontal line segment with the second end-point $p_3$.

All parts from $p_{2k}$ to $p_{2k+1}$, where $k = 2, \ldots, 506$, begin with a horizontal line segment having its second end-point located to the right of all parts constructed before (in order to avoid self-intersections). The segment is followed by similar arcs (1, 2, 3, 5) as in the above part and two segments (4 and 6) of lengths making the part to finish at $p_{2k+1}$.

All parts from $p_{2k+1}$ to $p_{2k}$, $k = 1011, \ldots, 507$, are constructed in the similar way, but are located to the left of the polygon vertices.

Finally, the part from $p_{1013}$ to $p_1$ consists of:

1) a horizontal line segment with its second end-point located to the left of all parts constructed before,

2) a special arc turning up,

3) a vertical line segment with its second end-point having $W$ of vertical change above $p_1$,

4) a special arc turning right,

5) a horizontal line segment with its second end-point having $2W$ of horizontal change to the left of $p_1$,

6,7) two special arcs turning down and right, just to $p_1$.

For illustration, see the figure below related to an 11-gon.