Problems for NCUMC 2024
21.04.2024

1. Find all continuous on $\mathbb{R}$ functions f which satisfy the relation $f(x+2024)=\frac{1+f(x)}{1-f(x)},\ \forall x\in\mathbb{R}$.

2. Polynomial $P(x)=x^{2024}+c_{2022}x^{2022}+c_{2021}x^{2021}+...+c_0$ has 2024 real roots $b_1<b_2<...<b_{2024}$. Let us construct the infinite sequence by repeating these numbers $b_1,b_2,...,b_{2024},b_1,b_2,...,b_{2024},b_1,b_2,...,b_{2024},...$ Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of roots of the equation $\sqrt{x}\sin x=1$ taken in increasing order. Does the series $\sum_{n=1}^{\infty} b_n \sin a_n$ converge?

3. Is it true that for any continuously differentiable on [0,1] function f the following inequality holds: $\left|f(\frac{1}{2})\right|\le\int_0^1|f(x)|\,dx+\frac{1}{2}\int_0^1|f'(x)|\,dx$?

4. Consider the two-dimensional dynamical system $\dot{x}=f(x)$ with polynomial right-hand side. Let $x_\varphi(t)$ be the solution to the system maximally extended to the right and satisfying the initial condition $x_\varphi(0)=(\cos\varphi,\sin\varphi)$. For all $\varphi\in(-\pi,\pi)$ the solution $x_\varphi(t)$ tends to 0 as $t\to+\infty$. Which options are possible for the solution $x_\pi(t)$:

a) not to tend to 0 as $t\to+\infty$;
b) be unbounded for $t\ge 0$;
c) be unextensible onto $[0,+\infty)$?

5. Let n be a given positive integer. Find the minimal *d* such that for all distinct complex numbers $z_1,...z_n$ there exists a complex polynomial $p(z)$ of degree *d* such that $|p(z_1)|>\max_{1<j\le n}|p(z_j)|$?

6. A sequence $0<a_1<a_2<...$ and positive number C are chosen so that $|e^{ia_1}+e^{ia_2}+...+e^{ia_n}|\le C$ for all positive integer n. Prove that $a_n\ge\frac{n}{2C}-2$ for all n.

7. Let the facets of an n-dimensional simplex be given by the equations $a_{i1}x_1+a_{i2}x_2+...a_{in}x_n+a_{i,n+1}=0, i=\overline{1,n+1}$. Prove that the volume of the simplex is $V_n=\frac{1}{n!}\left|\frac{\Delta^n}{\Delta_1\Delta_2...\Delta_{n+1}}\right|$, where $\Delta=\det(a_{ij})$, Δ_i is the algebraic complement of element $a_{i,n+1}, i=\overline{1,n+1}$.

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