1. Find all continuous on $\mathbb{R}$ functions f which satisfy the relation $f(x+2024)=\frac{1+f(x)}{1-f(x)}, \forall x\in\mathbb{R}$.

Solution.

Let us consider possible values of the function f. It is not possible that $f(x)=1$ because in this case $f(x+2024)$ does not defined.

If $f(x_1)=0$ then $f(x_1+2024)=1$ and $f(x_1+4048)$ does not defined.

If $0<f(x_1)<1$ then $f(x_1+2024)=\frac{1+f(x_1)}{1-f(x_1)}>1$. Hence, due to the continuity of f there exists a point x_2 between x_1 and x_1+2024 for which $f(x_2)=1$. But it is forbidden.

If $1<f(x_1)$ then $f(x_1+2024)=\frac{1+f(x_1)}{1-f(x_1)}<0$. Hence, due to the continuity of f there exists a point x_2 between x_1 and x_1+2024 for which $f(x_2)=1$. But it is forbidden.

If $-1\le f(x_1)<0$ then $f(x_1+2024)=\frac{1+f(x_1)}{1-f(x_1)}\ge 0$. Hence, due to the continuity of f there exists a point x_2 between x_1 and x_1+2024 for which $f(x_2)=0$. But it is forbidden.

If $f(x_1)<-1$ then $-1<f(x_1+2024)=\frac{1+f(x_1)}{1-f(x_1)}<0$. Hence, we obtain the previous case for x_1+2024. But it is forbidden.

Thus, the function f can not take any value. Such function does not exist.

2. Polynomial $P(x) = x^{2024} + c_{2022}x^{2022} + c_{2021}x^{2021} + ... + c_0$ has 2024 real roots $b_1 < b_2 < ... < b_{2024}$. Let us construct the infinite sequence by repeating these numbers $b_1, b_2, ..., b_{2024}, b_1, b_2, ..., b_{2024}, b_1, b_2, ..., b_{2024}, ...$ Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of roots of the equation $\sqrt{x}\sin x = 1$ taken in increasing order. Does the series $\sum_{n=1}^{\infty} b_n \sin a_n$ converge?

Answer. The series converges.

Solution.

Due to the Viete theorem, $\sum_{n=1}^{2024} b_n = 0$. Hence, partial sums of the series $\sum_{n=1}^{\infty} b_n$ are bounded

$$\left|\sum_{n=1}^{m} b_n\right| \le \sum_{n=1}^{2024} |b_n| = M .$$

Equation $\sqrt{x}\sin x = 1$ has two roots at each segment $[2\pi k, 2\pi k + \pi]$. Correspondingly, the sequence of roots $\{a_n\}_{n=1}^{\infty}$ monotonically tends to infinity. At the same time $\sin a_n = \frac{1}{\sqrt{a_n}} \to 0$ monotonically. Hence, due to the Dirichlet theorem the series $\sum_{n=1}^{\infty} b_n \sin a_n$ converges.

3. Is it true that for any continuously differentiable on [0,1] function f the following inequality holds: $\left|f(\frac{1}{2})\right| \le \int\limits_0^1 |f(x)|\,dx + \frac{1}{2}\int\limits_0^1 |f'(x)|\,dx$?

Solution.

$$f(\frac{1}{2}) = f(\frac{1}{2}) + \int\limits_0^1 f(x)dx - \int\limits_0^{1/2} f(x)dx - \int\limits_{1/2}^1 f(x)dx =$$

$$f(\frac{1}{2}) + \int\limits_0^1 f(x)dx - \frac{1}{2}(xf(x)\big|_0^{1/2} - \int\limits_0^{1/2} xf'(x)dx) -$$

$$\frac{1}{2}((x-1)f(x)\big|_{1/2}^{1} - \int\limits_{1/2}^1 (x-1)f'(x)dx) =$$

$$\int\limits_0^1 f(x)dx + \int\limits_0^{1/2} xf'(x)dx + \int\limits_0^{1/2} (x-1)f'(x)dx.$$

Hence,

$$\left|f(\frac{1}{2})\right| = \left|\int\limits_0^1 f(x)dx + \int\limits_0^{1/2} xf'(x)dx + \int\limits_{1/2}^1 (x-1)f'(x)dx\right| \le$$

$$\int\limits_0^1 |f(x)|dx + \int\limits_0^{1/2} |xf'(x)|dx + \int\limits_{1/2}^1 |(x-1)f'(x)|dx \le$$

$$\int\limits_0^1 |f(x)|dx + \frac{1}{2}\int\limits_0^{1/2} |f'(x)|dx + \frac{1}{2}\int\limits_{1/2}^1 |f'(x)|dx =$$

$$\int\limits_0^1 |f(x)|dx + \frac{1}{2}\int\limits_0^1 |f'(x)|dx.$$

The proof is finished.

Problem

Consider the two-dimensional dynamical system $\dot{x} = f(x)$ with polynomial right-hand side. Let $x_\varphi(t)$ be the solution to the system maximally extended to the right and satisfying the initial condition $x_\varphi(0) = (\cos\varphi, \sin\varphi)$. For all $\varphi \in (-\pi; \pi)$ the solution $x_\varphi(t)$ tends to 0 as $t \to +\infty$.

Is it possible for the solution $x_\pi(t)$:

a) not to tend to 0 as $t \to +\infty$;

b) be unbounded for $t \geqslant 0$;

c) be unextensible onto $[0; +\infty)$?

Answer

All possible: a) yes; b) yes; c) yes.

Solution

Treating 2-vectors as complex numbers, consider the equation $\dot{z} = -z^2$, which can be resolved explicitly. All its solutions (besides $z \equiv 0$) are defined as $z(t) = 1/(t + C)$ with $C = 1/z(0)$. So, $z_\varphi(t) = 1/(t + e^{-i\varphi})$. This is well defined on $[0; +\infty)$ and tends to 0 as $t \to +\infty$ whenever $\varphi \in (-\pi; \pi)$. However, the unbounded solution $z_\pi(t) = 1/(t-1)$ cannot be extended to the right of 1 and therefore has no limit at $+\infty$. (Though the same formula $1/(t-1)$ defines on $(1; +\infty)$ a solution tending to 0, but it is not z_π because of the initial conditions.)

1. Let n be a given positive integer. Find the minimal d such that for all distinct complex numbers $z_1, \ldots, z_n$ there exists a complex polynomial $p(z)$ of degree d such that $|p(z_1)| > \max_{1<j\leqslant n} |p(z_j)|$.

Solution.

For $d = n-1$. For such d take a polynomial $p(z) = \prod_{j=2}^{n}(z - z_j)$, for example. If $d < n-1$, consider the points $z_1 = 0$, $z_2, \ldots, z_n$ being all roots of unity of degree $n-1$. Note that $z_2^k + \ldots + z_n^k = 0$ for all $k = 1, 2, \ldots, n-2$ (that may be proved as follows: for $w = e^{2\pi i/(n-1)}$ two arrays $(z_2, \ldots, z_n)$ and $(wz_2, \ldots, wz_n)$ are both enumerations of the roots of unity of degree $n-1$, thus the sums of k-th powers of these two arrays are equal: $z_2^k + \ldots + z_n^k = w^k(z_2^k + \ldots + z_n^k)$, but $w^k \neq 1$, hence $z_2^k + \ldots + z_n^k = 0$.) Thus the equation $q(z_2) + \ldots + q(z_n) = (n-1)q(0)$ holds for $q(z) = z$, $q(z) = z^2$, …, $q(z) = z^{n-2}$. It also holds for $q(z) = 1$. Hence by linearity it holds for every polynomial of degree at most $n-2$, that is, if $\deg p(z) \leqslant n-2$ then $p(z_2) + \ldots + p(z_n) = (n-1)p(0) = (n-1)p(z_1)$. Now by triangle inequality we get $(n-1)|p(z_1)| \leqslant \sum_{k=2}^{n} |p(z_k)| \leqslant (n-1)\max_{2\leqslant k\leqslant n} |p(z_k)|$ — a contradiction.

2. A sequence $0 < a_1 < a_2 < a_3 < \ldots$ and positive number C are chosen so that

$$|e^{ia_1} + e^{ia_2} + \ldots + e^{ia_n}| \leqslant C$$

for all positive integer n. Prove that $a_n \geqslant \frac{n}{2C} - 2$ for all n.

Solution.

We need

Lemma. Assume that $0 < T < \pi$ and $a_m - a_s \leqslant T$ for certain indices $s \leqslant m$. Then

$$|e^{ia_s} + \ldots + e^{ia_m}| \geqslant (m - s + 1) \cos \frac{T}{2}$$

Proof of the lemma. Put $\theta = a_s + T/2$. Then $\theta - T/2 = a_s \leqslant a_m \leqslant \theta + T/2$, hence all $m-s+1$ numbers $b_s := a_s - \theta, \ldots, b_m := a_m - \theta$ belong to $[-T/2, T/2]$. Therefore

$$|e^{ia_s} + \ldots + e^{ia_m}| = |e^{ib_s} + \ldots + e^{ib_m}| \geqslant \Re(e^{ib_s} + \ldots + e^{ib_m}) \geqslant (m-s+1) \cos \frac{T}{2}.$$

Now, assuming the contrary to the claim of the problem, consider the minimal n for which $a_n < n/(2C) - 2$. Then in particular $0 < a_n < n/(2C) - 2$ and $n > 4C$, i.e., $n \geqslant k := \lceil 4C \rceil$. Consider the numbers $a_{n-k+1}, \ldots, a_n$. From minimality of n we have $a_{n-k+1} \geqslant (n - k + 1)/(2C) - 2$, therefore $a_n - a_{n-k+1} < (k-1)/(2C) < 2$. Lemma (for $T = 2$) yields

$$|e^{ia_{n-k+1}} + \ldots + e^{ia_n}| \geqslant k \cos 1 > k/2 \geqslant 2C,$$

contradicting to the given bound (by triangle inequality).

7. Let the facets of an n-dimensional simplex be given by the equations $a_{i1}x_1 + a_{i2}x_2 + ...a_{in}x_n + a_{i,n+1} = 0, i = \overline{1, n+1}$. Prove that the volume of the simplex is $V_n = \frac{1}{n!}\left|\frac{\Delta^n}{\Delta_1\Delta_2...\Delta_{n+1}}\right|$, where $\Delta = \det(a_{ij})$, Δ_i is the algebraic complement of element $a_{i,n+1}, i = \overline{1, n+1}$.

Solution

Lemma. For regular (non-degenerate) square matrix (a_{ij}) of order $(n+1)$, the following relation takes place

$$\det(\Delta_{ij}) = \det(a_{ij})^n,$$

where Δ_{ij} is the algebraic complement of element a_{ij}, $i, j = \overline{1, n+1}$.

Proof. Using the regularity of matrix (a_{ij}), one obtains:

$$1 = \det E = \det\left((a_{ij}) \cdot (a_{ij})^{-1}\right) = \det(a_{ij}) \cdot \det\left(\frac{\Delta_{ij}}{\det(a_{ij})}\right) =$$

$$= \det(a_{ij}) \cdot \frac{\det(\Delta_{ij})^T}{\det(a_{ij})^{n+1}} = \frac{\det(\Delta_{ij})}{\det(a_{ij})^n}.$$

This completes the proof:

$$\det(\Delta_{ij}) = \det(a_{ij})^n.$$

Let Γ_i be the facets of an n-dimensional simplex given by the equations $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + a_{i,n+1} = 0$, where $i = \overline{1, n+1}$, A_i be a vertex of the simplex for which the following condition holds

$$A_i = \Gamma_1 \cap \Gamma_2 \cap ... \cap \Gamma_{i-1} \cap \Gamma_{i+1} \cap ... \cap \Gamma_{n+1}$$

Particularly, for the vertex A_1 one has:

$$\begin{cases} a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + a_{2,n+1} = 0, \\ a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n + a_{3,n+1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n+1,1}x_1 + a_{n+1,2}x_2 + \cdots + a_{n+1,n}x_n + a_{n+1,n+1} = 0. \end{cases}$$

We know that the equations determine the simplex, hence the system has unique solution (coordinates of A_1), i.e. the system determinant is not zero. The coordinates of A_1 can be obtained by the Kramer formulas:

$$A_1\left(\frac{\Delta_{11}}{\Delta_{1,n+1}}, \frac{\Delta_{12}}{\Delta_{1,n+1}}, \dots, \frac{\Delta_{1n}}{\Delta_{1,n+1}}\right).$$

Analogously, one obtains the coordinates of A_i, where $i = \overline{1, n+1}$:

$$A_i\left(\frac{\Delta_{i1}}{\Delta_{i,n+1}}, \frac{\Delta_{i2}}{\Delta_{i,n+1}}, \dots, \frac{\Delta_{in}}{\Delta_{i,n+1}}\right).$$

Substituting the coordinates of A_i, $i = \overline{1, n+1}$, in the formula for the volume of n-dimensional simplex

$$V_n = \frac{1}{n!}\mathrm{mod}(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}, \dots, \overrightarrow{A_1A_n}),$$

one obtains:

$$V_n = \frac{1}{n!}\bmod \begin{vmatrix} \frac{\Delta_{21}}{\Delta_{2,n+1}} - \frac{\Delta_{11}}{\Delta_{1,n+1}} & \frac{\Delta_{22}}{\Delta_{2,n+1}} - \frac{\Delta_{12}}{\Delta_{1,n+1}} & \dots & \frac{\Delta_{2n}}{\Delta_{2,n+1}} - \frac{\Delta_{1n}}{\Delta_{1,n+1}} \\ \frac{\Delta_{31}}{\Delta_{3,n+1}} - \frac{\Delta_{11}}{\Delta_{1,n+1}} & \frac{\Delta_{32}}{\Delta_{3,n+1}} - \frac{\Delta_{12}}{\Delta_{1,n+1}} & \dots & \frac{\Delta_{3n}}{\Delta_{3,n+1}} - \frac{\Delta_{1n}}{\Delta_{1,n+1}} \\ \dots & \dots & \dots & \dots \\ \frac{\Delta_{n+1,1}}{\Delta_{n+1,n+1}} - \frac{\Delta_{11}}{\Delta_{1,n+1}} & \frac{\Delta_{n+1,2}}{\Delta_{n+1,n+1}} - \frac{\Delta_{12}}{\Delta_{1,n+1}} & \dots & \frac{\Delta_{n+1,n}}{\Delta_{n+1,n+1}} - \frac{\Delta_{1n}}{\Delta_{1,n+1}} \end{vmatrix}.$$

This determinant can be rewritten as $(n+1)$ –dimensional determinant:

$$V_n = \frac{1}{n!}\bmod \begin{vmatrix} \frac{\Delta_{11}}{\Delta_{1,n+1}} & \frac{\Delta_{12}}{\Delta_{1,n+1}} & 1 \\ \frac{\Delta_{21}}{\Delta_{2,n+1}} & \frac{\Delta_{22}}{\Delta_{2,n+1}} & 1 \\ \dots & \dots & \dots \\ \frac{\Delta_{n+1,1}}{\Delta_{n+1,n+1}} & \frac{\Delta_{n+1,2}}{\Delta_{n+1,n+1}} & 1 \end{vmatrix}$$

$$= \frac{1}{n!}\bmod \frac{1}{\Delta_{1,n+1}\Delta_{2,n+1}\dots\Delta_{n+1,n+1}} \begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{1,n+1} \\ \Delta_{21} & \Delta_{22} & \Delta_{2,n+1} \\ \dots & \dots & \dots \\ \Delta_{n+1,1} & \Delta_{n+1,2} & \Delta_{n+1,n+1} \end{vmatrix}$$

$$= \frac{1}{n!}\bmod \frac{\det(\Delta_{ij})}{\Delta_{1,n+1}\Delta_{2,n+1}\dots\Delta_{n+1,n+1}}$$

Using Lemma, one comes to the formula:

$$V_n = \frac{1}{n!}\bmod \frac{\det(\Delta_{ij})}{\Delta_{1,n+1}\Delta_{2,n+1}\dots\Delta_{n+1,n+1}} = \frac{1}{n!}\bmod \frac{\det(a_{ij})^n}{\Delta_{1,n+1}\Delta_{2,n+1}\dots\Delta_{n+1,n+1}}$$

Which gives us the result

$$V_n = \frac{1}{n!}\bmod \frac{\Delta^n}{\Delta_1\Delta_2\Delta_3\dots\Delta_{n+1}}.$$