Problems for NCUMC 2025.   20.04.2025

1.Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots \cos n^2$. For example, $d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$.

The argument of cos is always in radians, not degrees. Evaluate $\lim\limits_{n \to \infty} d_n$.

2.There are 10 people in a team, and for any two people their *compatibility* is defined which is a non-negative number. For a triple of members in a team, define the *coherence* of this triple as the product of the three pairwise compatibilities of the members of the triple. Find the largest possible value of the sum of coherences of all 120 triples, provided that the sum of squares of all 45 pairwise compatibilities is 45.

3: Let $f, g: [0,2025] \to R$ be differentiable functions such that $\int_0^{2025} f(x)dx = 0$. Prove that there is some $c \in (0,2025)$ satisfying

$$f'(c) \int_c^{2025} g(x)dx + g'(c) \int_c^{2025} f(x)dx = 2f(c)g(c).$$

4. The measure μ is given and finite on $[-1;1]$. Let $\hat{\mu}(z) = \int_{-1}^{1} \frac{d\mu(x)}{1-xz}$, $z \in D = \{z \in C: |z| < 1\}$. Any complex number can be written as $z = r \cdot e^{i\varphi}$, where $r \geq 0, \varphi \in [0,2\pi]$. Prove that $\int_0^{2\pi} \left|\hat{\mu}\left(re^{i\varphi}\right)\right|^p d\varphi < \infty$, $p \in (0,1)$.

5. Let a_n be the number of complex roots of the equation $z^n + 2z + 2 = 0$ lying in the disk $|z| < 1$. Are there such positive integer numbers m and k that $a_{n+m} = a_n + k$ for all sufficiently large n?

6. Does the equation $\frac{d^4 y}{dx^4} = y^{2025}$ have a solution defined on the whole real axis and not identically equal to zero?

7. Does there exist a continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f(x), f(x) + \pi, e - f(x), f(x) + x$ are irrational for all irrational x?

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