7. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x), f(x) + \pi, e - f(x), f(x) + x$$

are irrational for all irrational x?

Solution

7. Yes. The idea is to construct a strictly increasing continuous bijective function f such that for every rational q the value q of f, $f + \pi$, $e - f$ and $f(x) + x$ are obtained at rational points. To this goal, we enumerate all rational points as $q_1, q_2, \ldots$. After n-th step, our function is defined on a finite set $E_n \subset \mathbb{Q}$ (where $E_1 \subset E_2 \subset E_3 \ldots$, and once we defined $f(x)$ we do not change it on further steps); f strictly increases on E_n; both E_n and $f(E_n)$ contain all numbers of the form k/n for integers $k \in [-n, n]$. Also, there exist the elements a_n, b_n, c_n, d_n on E_n such that $f(a_n) = f(b_n) + \pi = e - f(c_n) = d_n + f(d_n) = q_n$. If we manage to satisfy these properties, then we determine the restriction of f on $\mathbb{Q}$ which defines f itself by continuity, and f enjoys the required properties. Now how to do the step number n. This step itself consists of adding one by one finitely many points $(x, f(x))$ to the graph of f. For example, to construct the point d_n for which $d_n + f(d_n) = q_n$, we find the consecutive numbers $v < u$ for which f is already defined and $v + f(v) \leqslant q_n \leqslant u + f(u)$ (put $u = +\infty$ if all already defined values of $x + f(x)$ exceed q_n). Then either $d_n = v$ or $d_n = u$ already works, or we may find d_n between v and u for which $d_n + f(v) < q_n < d_n + f(u)$ (take for d_n any rational number in the interval between $\max(v, q_n - f(u))$ and $\min(u, q_n - f(v))$) and put $f(d_n) := q_n - d_n$. Satisfying other conditions, including making E_n and $f(E_n)$ dense in the aforementioned sense, is obtained similarly.

Problem 1

Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \ldots, \cos n^2$. (For example, $d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$. The argument of cos is always in radians, not degrees.)

Evaluate $\lim_{n\to\infty} d_n$.

Solution

The limit is zero, because the determinant is zero for $n \geq 3$. Let D_n be the matrix that we're taking the determinant of. Let E_n be the matrix

$$E_n = \begin{bmatrix} e^{i} & e^{2i} & \cdots & e^{in} \\ e^{i(n+1)} & e^{i(n+2)} & \cdots & e^{2in} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i(n^2-n+1)} & e^{i(n^2-n+2)} & \cdots & e^{in^2} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{in} \\ \vdots \\ e^{i(n^2-n)} \end{bmatrix} \begin{bmatrix} e^{i} & e^{2i} & \cdots & e^{in} \end{bmatrix}$$

It is evident from the factorization shown that E_n has rank 1. So also does its complex conjugate $\overline{E_n}$ have rank 1. But then $D_n = \frac{1}{2}E_n + \frac{1}{2}\overline{E_n}$. The rank of a sum of matrices is less than or equal to the sum of the ranks, so $\operatorname{rank}(D_n) \leq 2$. Hence for $n \geq 3$, $\det(D_n) = 0$.

Remark: The terms in the second row have the form $\cos(n+k)$. Then if the second row is multiplied by $2\cos n$, we will get $2\cos n\cos(n+k) = \cos(2n+k)+\cos k$, which is the sum of the first row and the third row and we are done.

Problem 2

There are 10 people in a team, and for any two people. their *compatibility* is defined — a non-negative number. For a triple of members in a team, define the *coherence* of this triple as the product of the three pairwise compatibilities of the members of the triple. Find the largest possible value of the sum of coherences of all 120 triples, provided that the sum of squares of all 45 pairwise compatibilities is 45.

Solution.

The answer is 120. This value is obtained when all compatibilities are equal to 1. Let us prove that the sum of coherences is not greater than 120. Consider a symmetric matrix M of compatibilities (with zeros on the diagonal). Its trace is 0, and the trace of its square is equal to twice the sum of the squares compatibilities, i.e., 1. Note that the trace of the cube of the matrix M is equal to the sum of coherences of triples multiplied by 6. Thus, we need to prove that the trace of M^3 is at most 720. Let's denote the eigenvalues of the matrix M by $t_1, \ldots, t_{10}$. Then, $\sum t_i = 0$, $\sum t_i^2 = 90$. Suppose that $t_i > 9$ for some i. Then the sum of the remaining eigenvalues is less than -9, and therefore for the sum of their squares we have

$$\sum_{j \neq i} t_j^2 = \sum_{j \neq i} ((-1 - 2t_j) + (t_j + 1)^2) \geqslant -9 - 2 \sum_{j \neq i} t_j > 9,$$

also $t_i^2 > 81$ and the sum of all ten squares is greater than 90 — contradiction. So, $t_i \leqslant 9$ for all i, therefore

$$0 \geqslant \sum_i (t_i - 9)(t_i + 1)^2 = \sum_i t_i^3 - 7 \sum_i t_i^2 - 17 \sum_i t_i - 90 = \sum_i t_i^3 - 720,$$

as requested.

Problem: Let $f, g: [0,2025] \to R$ be differentiable functions such that $\int_0^{2025} f(x)dx = 0$. Prove that there is some $c \in (0,2025)$ satisfying

$$f'(c) \int_c^{2025} g(x)dx + g'(c) \int_c^{2025} f(x)dx = 2\, f(c)g(c).$$

Solution: Define

$$h(t) = (\int_0^t f(x)dx)\,(\int_{2025}^t g(x)dx)$$

f, g are differentiable, hence h is twice differentiable. We have $h(0) = h(2025) = 0$, thus by Rolle's Theorem there exists $\theta \in (0,2025)$ such that $h'(\theta) = 0$. Notice that

$$h'(t) = f(t) \int_{2025}^t g(x)dx + g(t) \int_0^t f(x)dx$$

and also by condition $\int_0^{2025} f(x)dx = 0$ we have $h'(2025) = 0 = h'(\theta)$, therefore by Rolle's Theorem we can find $c \in (\theta, 2025) \in (0,2025)$ such that $h''(c)=0$, that is

$$h''(c) = -f'(c) \int_c^{2025} g(x)dx + g(c)f(c) + g'(c) \int_0^c f(x)dx + g(c)f(c) =$$

$$= 2f(c)g(c) - f'(c) \int_c^{2025} g(x)dx + g'(c)(\int_0^{2025} f(x)dx$$

$$- \int_c^{2025} f(x)dx) = 2f(c)g(c) - f'(c) \int_c^{2025} g(x)dx - g'(c) \int_c^{2025} f(x)dx$$

Therefore $h''(c) = 0$ implies the

$$f'(c) \int_c^{2025} g(x)dx + g'(c) \int_c^{2025} f(x)dx = 2\, f(c)g(c).$$

The measure μ is given and finite on $[-1;1]$. Let

$$\hat{\mu}(z)=\int_{-1}^{1}\frac{d\mu(x)}{1-xz},\qquad z\in D=\{z\in\mathbb{C}:|z|<1\}.$$

Any complex number can be written as $z=r\cdot e^{i\varphi}$, where $r\geq 0,\varphi\in[0,2\pi]$. Prove that

$$\int_{0}^{2\pi}\left|\hat{\mu}\left(re^{i\varphi}\right)\right|^{p}d\varphi<\infty,\qquad p\in(0,1).$$

Solution. Let

$$\int_{-1}^{1}d\mu(x)=C.$$

If $|x|<0{,}5$, then $|1-xz|\geq 1-|x|\cdot|z|\geq 1-|x|\geq 0{,}5$. That is why

$$\int_{-1/2}^{1/2}\frac{d\mu(x)}{|1-xz|}\leq 2C.\ \ (1)$$

Next, if $z=re^{i\varphi}\in D$ and $|x|\geq 0{,}5$ we will have

$$|1-xz|=\sqrt{(1-xr\cos\varphi)^2+(xr\sin\varphi)^2}\geq|x|\cdot r\cdot|\sin\varphi|\geq\frac{r}{2}\cdot|\sin\varphi|.$$

Therefore

$$\int_{-1}^{-1/2}\frac{d\mu(x)}{|1-xz|}+\int_{1/2}^{1}\frac{d\mu(x)}{|1-xz|}\leq\frac{2}{r\cdot|\sin\varphi|}\cdot C\,.\ \ (2)$$

From (1) and (2) it follows that

$$|\hat{\mu}(z)|\leq\int_{-1}^{1}\frac{d\mu(x)}{|1-xz|}\leq\frac{4C}{r\cdot|\sin\varphi|}.$$

We also know that

$$\sin x\geq\frac{2}{\pi}\cdot x,\qquad x\in\left[0,\frac{\pi}{2}\right].$$

That is why

$$\int_{0}^{2\pi}\left|\hat{\mu}\left(re^{i\varphi}\right)\right|^{p}d\varphi\leq\left(\frac{4C}{r}\right)^{p}\cdot\int_{0}^{2\pi}\frac{d\varphi}{|\sin\varphi|^{p}}=4\cdot\left(\frac{4C}{r}\right)^{p}\cdot\int_{0}^{\frac{\pi}{2}}\frac{d\varphi}{|\sin\varphi|^{p}}\leq$$

$$\leq 4\cdot\left(\frac{2\pi C}{r}\right)^{p}\cdot\int_{0}^{\frac{\pi}{2}}\frac{d\varphi}{\varphi^{p}}=4\cdot\left(\frac{2\pi C}{r}\right)^{p}\cdot\frac{\varphi^{1-p}}{1-p}\Bigg|_{0}^{\frac{\pi}{2}}=\frac{2\pi}{1-p}\cdot\frac{(4C)^{p}}{r^{p}}<\infty,\qquad p\in(0,1).\ \blacksquare$$

Problem 5.

Let a_n be the number of complex roots of the equation $z^n + 2z + 2 = 0$ lying in the disk $|z| \leqslant 1$. Are there such positive integer numbers m and k that $a_{n+m} = a_n + k$ for all sufficiently large n?

Solution

5. No. First, note that all roots of the polynomial $p_n(z) = z^n + 2z + 2$ in the unit disk are simple (not multiple): indeed, $p'_n(z) = nz^{n-1} + 2$, so, if $p_n(z) = p'_n(z) = 0$, then $0 = np_n(z) - zp'_n(z) = (2n-2)z + 2n$, which is not 0 in the unit disk.

Further, there are two points α and $\bar{\alpha}$ on the circle for which $|P(\alpha)| = |P(\bar{\alpha})| = 1$, where $P(z) = 2z + 2$.

Note that α is not a root of unity. Indeed, α and $\bar{\alpha}$ are the roots of the equation $(2z+2)(2/z+2) = 1$, so $z+1/z = -7/4$. Note that if $z+1/z = a/2^m$ with odd a and natural m, then $z^2 + 1/z^2 = (z+1/z)^2 - 2 = a^2/2^{2m} - 2$ is an irreducible fraction with denominator 2^{2m}. Continuing in this fashion, we see that all the numbers $\alpha + 1/\alpha$, $\alpha^2 + 1/\alpha^2$, $\alpha^4 + 1/\alpha^4$, $\ldots$ are pairwise distinct, which is impossible if α is a root of unity.

The points α and $\bar{\alpha}$ divide the unit circle into arcs D_+ and D_-, on D_+ we have $|P(z)| > 1$, on D_- we have $|P(z)| < 1$. Let's count the number of roots $z^n + P(z)$ in the unit disk using the argument principle. On D_+ we have $z^n + P(z) = P(z)(1 + z^n/P(z))$, the second multiplier always lies in the right half-plane, so that the change of of its argument as we move along D_+ is bounded. The change of the argument of $P(z)$ as we move along D_+ is also bounded (it does not depend on n). On D_- we write $z^n + P(z) = z^n(1 + P(z)/z^n)$. The second factor is again in the right half-plane, so, the the change of its argument is bounded, and the change of the argument of the first factor equals $\theta \cdot n$, where θ is the length of D_-. But θ/π is irrational (as α is not a root of unity), thus, the limit $\lim a_n/n$ eists and is irrational. While, if $a_{n+k} = a_n + m$ for large enough n, then $\lim a_n/n = m/k$.

Problem 6 Does the equation $\frac{d^4y}{dx^4} = y^{2025}$ have a solution defined on the whole real axis and not identically equal to zero?

Solution. Let $y_a(x)$ be the maximally extended solution to the equation with initial values

$$y_a(0) = 1,\ y_a'(0) = 0,\ y_a''(0) = a,\ y_a'''(0) = 0.$$

Let a^* be the supremum of the set A consisting of all $a \in \mathbb{R}$ with the related solution $y_a(x)$ having at least one zero. If $a_1 \in A$ and $a_2 < a_1$, then monotonic considerations yield $a_2 \in A$.

Let $\xi(a)$ be the minimal $x > 0$ such that $y_a(x) = 0$. The implicit function theorem provides continuity of ξ on $(-\infty; a^*)$. To apply this theorem we need the condition $y_a'(\xi(a)) \neq 0$. But, if it does not hold for some $a < a^*$, i. e. $y_a(\xi(a)) = y_a'(\xi(a)) = 0$, then, due to the inequality $y_a'''(\xi(a)) = y_a'''(0) + \int_0^{\xi(a)} y(s)^{2025} ds > 0$, we obtain $y_a''(\xi(a)) \geqslant 0$. Really, if $y_a''(\xi(a)) < 0$, then $y_a(x) < 0$ at some $x \in (0; \xi(a))$, which contradicts to the definition of $\xi(a)$. But, if $y_a''(\xi(a)) \geq 0$, then any small increase of a produces a solution, which by monotonic considerations must be positive at some point, as well as its derivatives y', y'', y''', and therefore never vanishes. This is possible only if $a = a^*$.

Thus, the function $\xi(a)$ is continuous as well as $y_a''(\xi(a))$. The last one is negative for $a < 0$ with sufficiently large absolute value and positive for $a \in A$ sufficiently close to a^*. So, there exists $b \in A$ such that $y_b''(\xi(b)) = 0$.

Note that for our equation, the reflections $x \mapsto -x$ and $y \mapsto -y$ as well as translations along the x-axis transform any solution to another one. So, the solution y_b defined on the segment $[0; \xi(b)]$ can be extended onto the segment $[-\xi(b); 0]$ by the horizontal reflection and onto the segment $[\xi(b); 2\xi(b)]$ by the rotation around the point $(\xi(b), 0)$. Extending the solution in both directions on and on, we obtain a periodic solution defined on the whole real axis.